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Infinitesimal Orbits Near Any Point on Fawzy Equilibrium Circle in the Three Dimensional Restricted Three Body Problem

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Abstract

In the present work the three dimensional restricted three body problem (in brief TDRTBP) is formulated. The relative equilibria are obtained. The location of Fawzy equilibrium circle is discovered (in brief FEC). The Hamiltonian near any point on the circumference of FEC is constructed. The infinitesimal orbits near FEC are derived using an approach developed by Delva and Hanslmeier. Implicit formulas for the position and momentum vectors for n$^{\underline{th}}$ order and explicit formulas for the the same vectors for the first, second and third orders are obtained.

Keywords: Restricted Three Body Problem, Fawzy Equilibruim Circle.

Introduction

The three-body problem is a very rich dynamical system in mathematical intricacy and practical applicability. The restricted three-body problem interested in the motion of a particle of negligible mass in the presence of two massive bodies. The subject of the periodic solutions of the restricted problem of three bodies acquired great importance and interest since the last decades of the 20th century. The main reason is the need for space mission orbits in the vicinity of one of the colinear or triangular libration points. There are five equilibrium points in the classical three body problem, namely L_i, $i = 1, 2, 3, 4, 5$, these points usually called Lagrangian or libration points. Abd El-Salam (2012) discovered the so called Fawzy $\xi\zeta$ - triangular equilibrium points in the plane $\eta = 0$; the peripindicular plane to the plane of motion of the primaries. The author also discovered the so called Fawzy equilibrium circle in the three body problem. This circle passes through the Lagrange's and Fawzy tringular points.

The literature is rich and the works dealing with the periodic orbits near the equilibrium points cannot be exhaustively reviewed. However, it will be beneficial to sketch some of these most important works. An early and very useful analysis of the behaviuor of the bodies near libration points has been compiled by a group of eminent scientists in a work by Duncombe and Szebehely (1966). Richardson (1980), Barden and Howell (1998), Howell (2001), Gomez *et al.* (1998; 2004), Selaru and Dimitrescu (1995), Namouni and Murray (2000), Corbera and Llibre (2003), Ashraf Hamdy *et al.* (2005).

The motivation of the present work is to study the the infinitesimal orbits near any point on the circumference of Fawzy equilibrium circle (The recently discovered equilibruim circle). A similar treatment has appeared in the work by Ashraf Hamdy *et al.* (2005), but they has applied the Delva-Hanslmeier technique for the infinitesimal orbits near the Lagrange's equilibruim points.

Three Dimensional RTBP

Let $\mathbf{r}_1, \mathbf{r}_2 \in \mathrm{R}^3$ being the positions of the two massive bodies 1 and 2 respectively, which is the solution of the two-body problem. Let the first mass be $\mu \in \left]0,1\right[$, thus the second mass has $1-\mu$ and the gravitational constant is equal to 1. Let us denote the center of mass $\mathbf{R} = \mu\mathbf{r}_1 + (1-\mu)\mathbf{r}_2$ one can easily prove that $\frac{d^2\mathbf{R}}{dt^2} = 0$. Transforming to co-moving coordinates $\bar{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{R},\ i = 1,2$ and let $\mathbf{r} \in \mathrm{R}^3$ denotes the position of the test particle, these yield the equations of motions in these new variables as;

$$\frac{d^2\bar{\mathbf{r}}}{dt^2} = -\frac{\mu}{\left\|\bar{\mathbf{r}}-\bar{\mathbf{r}}_1\right\|^3}\left(\bar{\mathbf{r}}-\bar{\mathbf{r}}_1\right) - \frac{(1-\mu)}{\left\|\bar{\mathbf{r}}-\bar{\mathbf{r}}_2\right\|^3}\left(\bar{\mathbf{r}}-\bar{\mathbf{r}}_2\right) \tag{1}$$

At this point we start making the following assumptions: (i) The primaries move in a circular plannar orbit around their center of masss with constant angular velocity normalized to 1 without loss of generality. (ii) The frame of reference is rotating with the rotation matrix defined below. In this coordinate system the primaries become stationary and lie along x_1-axis. (iii) The test particle moves in a plane perpindicular to the primaries'plane, with coordinates $\mathbf{x}^T = (x_1,\ x_2,\ x_3)^T$. Therefore we set,

$$\bar{\mathbf{r}} = \mathrm{R}(t)\left[x_1 \quad x_2 \quad x_3\right]^T,$$

$$\bar{\mathbf{r}}_1 = \mathrm{R}(t)\left[1-\mu \quad 0 \quad 0\right]^T,$$

$$\bar{\mathbf{r}}_2 = \mathrm{R}(t)\left[-\mu \quad 0 \quad 0\right]^T$$

in which $\mathrm{R}(t)$ is the rotation matrix, given below, with the following properties,

$$\mathrm{R}(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{d^2\mathrm{R}(t)}{dt^2} = -\mathrm{R}(t),\ \mathrm{R}^{-1}(t)\frac{d\,\mathrm{R}(t)}{dt} = J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can deduce the equations of motion for $\mathbf{x} \in \mathrm{R}^3$:

$$\frac{d^2\mathbf{x}}{dt^2} - \mathbf{x} + 2J\frac{d\mathbf{x}}{dt} = -\frac{\mu\left(\mathbf{x}-\left[1-\mu \quad 0 \quad 0\right]^T\right)}{\left\|\mathbf{x}-\left[1-\mu \quad 0 \quad 0\right]^T\right\|^3} - \frac{(1-\mu)\left(\mathbf{x}-\left[-\mu \quad 0 \quad 0\right]^T\right)}{\left\|\mathbf{x}-\left[-\mu \quad 0 \quad 0\right]^T\right\|^3}$$

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Finally, setting $\mathbf{p} = \frac{d\mathbf{x}}{dt} + J\mathbf{x}$ we find that these are the Hamiltonian equations of motion on $\mathbb{R}^6 / \left\{ \mathbf{x} = \begin{bmatrix} 1-\mu & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -\mu & 0 & 0 \end{bmatrix}^T \right\}$ with Hamiltonian

$$\mathrm{H} = \frac{1}{2}\sum_{i=1}^{3}(\mathbf{p}\,.\,\mathbf{p}) - \left(\xi p_\eta - \eta p_\xi\right) - \frac{\mu}{\left\| \mathbf{x} - \begin{bmatrix} 1-\mu & 0 & 0 \end{bmatrix}^T \right\|} - \frac{1-\mu}{\left\| \mathbf{x} - \begin{bmatrix} -\mu & 0 & 0 \end{bmatrix}^T \right\|} \tag{2}$$

where we have equipped $\mathbb{R}^6$ with the canonical symplectic form $d\xi \wedge dp_\xi + d\eta \wedge dp_\eta$, i.e. the equations of motion are given by

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathrm{H}}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial \mathrm{H}}{\partial \mathbf{x}}.$$

Relative Equilibria

The relative equilibria are the the solutions of the Hamiltonian vector field given by (2). Let us write the gravitational potential energy function as

$$\Omega(\mathbf{x}) = -\frac{\mu}{\left\| \mathbf{x} - \begin{bmatrix} 1-\mu & 0 & 0 \end{bmatrix}^T \right\|} - \frac{\mu - 1}{\left\| \mathbf{x} - \begin{bmatrix} -\mu & 0 & 0 \end{bmatrix}^T \right\|} \tag{3}$$

The equilibrium solutions of (2) are obtained setting all the partial derivatives of H equal to zero as;

$$\begin{aligned} \frac{\partial \mathrm{H}}{\partial p_{x_1}} &= p_{x_1} + x_2 = 0 & \qquad \frac{\partial \mathrm{H}}{\partial x_1} &= -p_{x_2} + \frac{\partial U(\mathbf{x})}{\partial x_1} = 0 \\ \frac{\partial \mathrm{H}}{\partial p_{x_2}} &= p_{x_2} - x_1 = 0 & \qquad \frac{\partial \mathrm{H}}{\partial x_2} &= p_{x_1} + \frac{\partial U(\mathbf{x})}{\partial x_2} = 0 \\ \frac{\partial \mathrm{H}}{\partial p_{x_3}} &= p_{x_3} = 0 & \qquad \frac{\partial \mathrm{H}}{\partial x_3} &= \frac{\partial U(\mathbf{x})}{\partial x_3} = 0 \end{aligned}$$

or equivalently

$$\frac{\partial \Omega(\mathbf{x})}{\partial x_1} = x_1, \qquad \frac{\partial \Omega(\mathbf{x})}{\partial x_2} = x_2, \qquad \frac{\partial \Omega(\mathbf{x})}{\partial x_3} = 0 \tag{4}$$

where $\mathbf{p}$ at the equilibrium point can easily be found once we solved (4) for $\mathbf{x}$ at the equilibrium point. Note that $\mathbf{x}$ solves (4) if and only if $\mathbf{x}$ is a stationary point of the function $U(\mathbf{x}) = \frac{1}{2}\left(x_1^2 + x_2^2\right) - \Omega(\mathbf{x})$ called the amended potential. Let us first look for equilibrium points of the amended potential that lie on the line of syzygy (The line on which all the three bodies similtanuously lie at some specified date), *i.e.* for which $x_2 = 0$. Note that $\frac{\partial U(\mathbf{x})}{\partial x_2} = 0$ is automatically satisfied in this case since $\left.\frac{\partial \Omega(\mathbf{x})}{\partial x_2}\right|_{x_2=0} = 0$, $\frac{\partial U(\mathbf{x})}{\partial x_1} = 0$ reduces to

$$\frac{d}{dx_1} U(x_1, 0) = \frac{d}{dx_1}\left(\frac{1}{2} x_1^2 + \frac{\mu}{|x_1 - 1 + \mu|} + \frac{1-\mu}{|x_1 + \mu|}\right) = 0 \tag{5}$$

Clearly,

$$\lim_{x_1 \to -\infty} U(x_1, 0) \to \infty, \qquad \lim_{x_1 \to -\mu} U(x_1, 0) \to \infty,$$

$$\lim_{x_1 \to 1-\mu} U(x_1, 0) \to \infty, \qquad \lim_{x_1 \to \infty} U(x_1, 0) \to \infty$$

so $U(x_1, 0)$ has at least one critical point on each of the intervals

$$]-\infty, -\mu[, \qquad]-\mu, 1-\mu[, \qquad]1-\mu, \infty[$$

To explore the concavity of the mentioned intervals, we calculate the second derivative as;

$$\frac{d^2}{dx_1^2} U(x_1, 0) = 1 + 2\frac{\mu}{|x_1 - 1 + \mu|^3} + 2\frac{\mu - 1}{|x_1 + \mu|^3} > 0 \tag{6}$$

So $U(x_1, 0)$ is convex on each of these intervals and we conclude that there is exactly one critical point in each of the intervals. The three relative equilibria on the line of syzygy are called the Eulerian equilibria. They are denoted by L_1; L_2 and L_3, where $L_1 \in]-\infty, -\mu[\times \{0\}$, $L_2 \in]-\mu, 1-\mu[\times \{0\}$ and $L_3 \in]1-\mu, \infty[\times \{0\}$.

Now we shall look for new equilibrium points that do not lie on the line of syzyg. Let us use

$$\left.\begin{aligned} \xi &= \sqrt{(x_1 + \mu - 1)^2 + x_2^2 + x_3^2}, \\ \eta &= \sqrt{(x_1 + \mu)^2 + x_2^2 + x_3^2}, \\ \zeta &= \sqrt{x_1^2 + x_2^2 + x_3^2} \end{aligned}\right\} \tag{7}$$

as coordinates in each of the half-planes $x_2 > 0$ and $x_2 < 0$. Then U can be written as

$$U = \frac{1}{2}(1-\mu)\left(\eta_2^2 + \frac{2}{\eta_2}\right) + \frac{1}{2}\mu\left(\xi_1^2 + \frac{2}{\xi_1}\right) - \frac{1}{2}\mu(1-\mu) \tag{8}$$

At the equilibrium points $\frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial \eta} = \frac{\partial U}{\partial \eta} = 0$

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$$\frac{\partial U}{\partial \xi} = \mu\left(\xi - \frac{1}{\xi^2}\right) = 0 \quad \Rightarrow \quad \xi = 1,$$
$$\frac{\partial U}{\partial \eta} = -(\mu - 1)\left(\eta - \frac{1}{\eta^2}\right) = 0 \quad \Rightarrow \quad \eta = 1$$

Substituting $\xi = 1, \quad \eta = 1$ back to the first two equations of (7) yields

$$\left.\begin{array}{l}(x_1 + \mu - 1)^2 + x_2^2 + x_3^2 = 1 \\ (x_1 + \mu)^2 + x_2^2 + x_3^2 = 1\end{array}\right\} \tag{9}$$

We can solve this system of two equations in three variables yields two variables namely; x_1 which will have a fixed value, and x_2 which will have a wide range as we shall see below. The third variable x_3 can be expreesed in terms x_2.
Subtracting the two equations of (9) yields

$$(x_1 + \mu - 1)^2 - (x_1 + \mu)^2 = 0 \;\Rightarrow\; (x_1 + \mu - 1) = (x_1 + \mu) \;\Rightarrow\; x_1 = \frac{1}{2} - \mu$$

Substituting $x_1 = \frac{1}{2} - \mu$ back into any equation of (7) we get

$$x_2^2 + x_3^2 = \frac{3}{4} \quad \Rightarrow \quad x_3 = \pm\sqrt{\frac{3}{4} - x_2^2}$$

provided that

$$\|x_2\| \le \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \frac{-\sqrt{3}}{2} \le x_2 \le \frac{\sqrt{3}}{2}.$$

Therefore we have infinite number of solutions (including Lagrange triangular equilibrium points and Fawzy triangular equilibrium points), their coordinates are given by

$$(x_1, \quad x_2, \quad x_3) = \left(\frac{1}{2} - \mu, \quad \frac{-\sqrt{3}}{2} \le x_2 \le \frac{\sqrt{3}}{2} \le, \quad x_3 = \pm\sqrt{\frac{3}{4} - x_2^2}\right) \tag{10}$$

Transformed Hamiltonian Near FEC

At this point we are interested in the infintesimal orbits near any point on the circumference of FEC. Moving the origin to any point on FEC and denoting to the new coordinates and momenta by $(X_1, X_2, X_3, P_{X_1}, P_{X_2}, P_{X_3})$ by the following substitution

$$\xi = X_1 + x_1, \;\; \eta = X_2 + x_2, \;\; \zeta = X_3 + x_3, \;\; p_\xi = P_{X_1} - x_2, \;\; p_\eta = P_{X_2} + x_1,$$

where

$$\left(x_1,\quad x_2,\quad x_3\right)=\left(\frac{1}{2}-\mu,\quad \frac{-\sqrt{3}}{2}\le x_2\le\frac{\sqrt{3}}{2}\le,\quad x_3=\pm\sqrt{\frac{3}{4}-x_2^2}\right).$$

These settings yields the transformed Hamiltonian as;

$$\begin{aligned}\mathrm{H}=&\frac{1}{2}\left(P_{X_1}^2+P_{X_2}^2\right)+\left(X_2P_{X_1}-X_1P_{X_2}\right)+\frac{1}{8}X_1^2+\left(\frac{1}{2}-\frac{3}{2}x_2^2\right)X_2^2+\left(\frac{1}{2}-\frac{3}{2}x_3^2\right)X_3^2\\&+\left(\frac{1}{2}-\mu\right)X_1+x_2X_2+x_3X_3-3\left(\frac{1}{2}-\mu\right)x_2X_1X_2-3\left(\frac{1}{2}-\mu\right)x_3X_1X_3\\&-3x_2x_3X_2X_3-\frac{9}{8}-\frac{1}{2}x_2^2\left(1-\frac{9}{32}\mu\right)+\frac{3}{8}\mu x_2^4\left(1-2x_2^2\right)+\frac{1}{2}\mu\left(1-\mu\right)\end{aligned}\tag{11}$$

Perturbation Approach and Solutions

We use an approach developed by Delva (1984) and Hanslmeier (1984) in which the procedure can be performed with a differential operator D . A special linear operator, the Lie operator, produces a Lie series. The convergence of the series is the same as for Taylor series, since the Lie series is only another form of the Taylor series whose terms are generated by the Lie operator. We will use this Lie series form for two reasons. The first is the requirement to build up a perturbative scheme at different orders of the orbital elements. The second is its usefulness in treating the non-autonomous system of differential equations and non-canonical systems . This enables a rapid successive calculation of the orbit. In addition we can change the stepsize easily (if necessary). This is an important advantage for the treatment of the problems which has a variable stepsize, *e.g.* for the mass change of the primaries. The formulas has an easy analytical structure and may be programmed without difficulty and without imposing extra conditions on the convergence. Since any desired number of terms can be found by iteration, the series can be continued up to any satisfactory convergence reached.

$$D\Xi=\sum_{i=1}^{3}\left(\frac{\partial\Xi}{\partial X_i}\frac{dX_i}{dt}+\frac{\partial\Xi}{\partial P_{X_i}}\frac{dP_{X_i}}{dt}\right)+\frac{\partial}{\partial t},\qquad\left(\Xi=\mathbf{X}\left(X_i,P_{X_i}\right),\mathbf{P}\left(X_i,P_{X_i}\right)\right)\tag{12}$$

Using Leibnitz formula for the n^{th} derivative of a product , namely

$$\frac{d^n}{dZ^n}\left[g(z)h(z)\right]\quad=\quad\sum_{m=0}^{n}C_n^m\frac{d^mg}{dZ^m}\frac{d^{n-m}h}{dZ^{n-m}},\qquad C_m^n=\frac{n!}{m!(n-m)!}$$

yields the n^{th} application of the Lie operator denoted by $D^{(n)}$ as;

$$D^{(n)}\Xi=\sum_{i=1}^{3}\sum_{m=0}^{n}C_n^m\left[\left(\frac{\partial^m\Xi}{\partial X_i^m}\frac{d^{n-m}X_i}{dt^{n-m}}+\frac{\partial^m\Xi}{\partial P_{X_i}^m}\frac{d^{n-m}P_{X_i}}{dt^{n-m}}\right)+\frac{\partial^n\Xi}{\partial t^n}\right]\tag{13}$$

Let $\mathrm{H}(X_1,X_2,X_3,P_{X_1},P_{X_2},P_{X_3})$ be the Hamiltonian function near any point on the circumference of Fawzy equilibrium circle as given by (11), and using the canonical

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equations of motion $\frac{dX_i}{dt}=\frac{\partial H}{\partial P_{X_i}}$, $\frac{dP_{X_i}}{dt}=-\frac{\partial H}{\partial X_i}$ to evaluate the derivatives $\frac{d^{n-m}X_i}{dt^{n-m}}$, $\frac{d^{n-m}P_{X_i}}{dt^{n-m}}$ then we can reach to the solutions (coordinate and momentum vectors, $\mathbf{X}$, $\mathbf{P}$ respectively) as;

$$\mathbf{X}=\left(e^{(t-t_\circ)D}\right)\mathbf{X}\Big|_{\mathbf{X}=\mathbf{X}_0}=\sum_{j=0}^{\infty}D^j\mathbf{X}\Big|_{\mathbf{X}=\mathbf{X}_0,\mathbf{P}=\mathbf{P}_0}\frac{(t-t_\circ)^j}{j\,!}=\sum_{j=0}^{\infty}\left\{D^{(n)}X_1+D^{(n)}X_2\right\}_{\mathbf{X}=\mathbf{X}_0,\mathbf{P}=\mathbf{P}_0}\frac{(t-t_\circ)^j}{j\,!} \quad (14)$$

$$\mathbf{P}=\left(e^{(t-t_\circ)D}\right)\mathbf{P}\Big|_{\mathbf{X}=\mathbf{X}_0,\mathbf{P}=\mathbf{P}_0}=\sum_{j=0}^{\infty}D^j\mathbf{P}\Big|_{\mathbf{X}=\mathbf{X}_0,\mathbf{P}=\mathbf{P}_0}\frac{(t-t_\circ)^j}{j\,!}=\sum_{j=0}^{\infty}\left\{D^{(n)}P_{X_1}+D^{(n)}P_{X_2}+D^{(n)}P_{X_3}\right\}_{\mathbf{X}=\mathbf{X}_0,\mathbf{P}=\mathbf{P}_0}\frac{(t-t_\circ)^j}{j\,!} \quad (15)$$

Solutions at Different Orders

In this section we are going to evaluate the solutions at different orders. From the definition of operator $D^{(n)}$, given in (13), we get the following expressions for the coefficients.

The First Order Solution

Setting $n=1$ we have the required coefficients in the equations (14), (15) to yield the first order solution as;

$$\begin{aligned}
D^{(1)}X_1 &= P_{X_1}-X_2,\\
D^{(1)}X_2 &= P_{X_2}+X_1,\\
D^{(1)}P_{X_1} &= P_{X_2}+\left(\frac{3}{2}x_2-3\mu x_2-\frac{1}{4}\right)X_1+3\left(\frac{1}{2}-\mu\right)x_3X_3-\left(\frac{1}{2}-\mu\right),\\
D^{(1)}P_{X_2} &= -P_{X_1}-2\left(\frac{1}{2}-\frac{3}{2}x_2^2\right)X_2-x_2+3\left(\frac{1}{2}-\mu\right)x_2X_1+3x_2x_3X_3,\\
D^{(1)}P_{X_3} &= -3\left(\frac{1}{2}-\mu\right)x_3X_1+3x_2x_3X_2-\left(1-3x_3^2\right)X_3-x_3
\end{aligned}$$

The Second Order Solution

Setting $n=2$ we have the required coefficients in the equations (14), (15) to yield the second order solution as;

$$\begin{aligned}
D^{(2)}X_1 &= 2P_{X_2}+\left(\frac{3}{2}x_2-3\mu x_2-\frac{5}{4}\right)X_1+3x_3\left(\frac{1}{2}-\mu\right)X_3-\left(\frac{1}{2}-\mu\right),\\
D^{(2)}X_2 &= 3\left(\frac{1}{2}-\mu\right)x_2X_1-3\left(1-x_2^2\right)X_2+3x_2x_3X_3-2P_{X_1}-x_2,\\
D^{(2)}P_{X_1} &= \left(\frac{3}{2}x_2-3\mu x_2-\frac{5}{4}\right)P_{X_1}+3\left(\frac{1}{2}-\mu\right)x_2X_1-\left(3\mu x_2-3x_2^2-\frac{3}{2}x_2+\frac{5}{4}\right)X_2\\
&\quad+3x_2x_3X_3+3\left(\frac{1}{2}-\mu\right)x_3-x_2,
\end{aligned}$$

$$D^{(2)}P_{X_2} = 3x_2\left(\frac{1}{2}-\mu\right)P_{X_1} - \left(2-3x_2^2\right)P_{X_2} + \left(3\mu x_2 - 3x_2^2 - \frac{3}{2}x_2 + \frac{5}{4}\right)X_1 + 3x_2\left(\frac{1}{2}-\mu\right)X_2$$
$$-3x_3\left(\frac{1}{2}-\mu\right)X_3 + \left(\frac{1}{2}-\mu\right),$$
$$D^{(2)}P_{X_3} = -3x_3\left(\frac{1}{2}-\mu\right)P_{X_1} + 3x_2x_3P_{X_2} - 3x_2x_3X_1 - 3x_3\left(\frac{1}{2}-\mu\right)X_2$$

The Third Order Solution

Setting $n=3$ we have the required coefficients in the equations (14), (15) to yield the second order solution as;

$$D^{(3)}X_1 = \left(\frac{13}{4}+\frac{3}{2}x_2 - 3x_2\mu\right)P_{X_1} + 6\left(\frac{1}{2}-\mu\right)x_2X_1$$
$$-\left(\frac{13}{4} - 6x_2^2 + \frac{3}{2}x_2 - 3\mu x_2\right)X_2 + 6x_2x_3X_3 - 2x_2,$$
$$D^{(3)}X_2 = \left(\frac{1}{2}+6\mu - 3x_2^2\right)x_2X_1 + 3\left(\frac{1}{2}-\mu\right)x_2X_2 - 3(1-2\mu)x_3X_3$$
$$+3\left(\frac{1}{2}-\mu\right)x_2P_{X_1} - \left(1-3x_2^2\right)P_{X_2} + (1-2\mu),$$
$$D^{(3)}P_{X_1} = 3x_2\left(\frac{1}{2}-\mu\right)P_{X_1} + \left(3x_2 + 3x_2^2 - 6\mu x_2 - \frac{5}{2}\right)P_{X_2}$$
$$+\left(9\mu^2x_2^2 - 9\mu x_2^2 + \frac{15}{2}\mu x_2 - \frac{3}{4}x_2^2 - \frac{15}{4}x_2 + \frac{25}{16}\right)X_1$$
$$+ \quad 3x_2\left(\frac{1}{2}-\mu\right)X_2 + 3\left(\mu - \frac{1}{2}\right)\left(3\mu x_2 - \frac{3}{2}x_2 + \frac{5}{4}\right)x_3X_3$$
$$-\left(\mu-\frac{1}{2}\right)\left(3\mu x_2 - \frac{3}{2}x_2 + \frac{5}{4}\right),$$
$$D^{(3)}P_{X_2} = \left(3\mu x_2 - 6x_2^2 - \frac{3}{2}x_2 + \frac{13}{4}\right)P_{X_1} + 3x_2(1-2\mu)P_{X_2}$$
$$-\frac{3}{8}x_2(2\mu-1)\left(6x_2 + 12x_2^2 - 12\mu x_2 - 5\right)X_1 + \left(9x_2^4 - 12x_2^2 - \frac{3}{2}x_2 + 3\mu x_2 + \frac{13}{4}\right)X_2$$
$$+x_2x_3\left(9\mu^2 - 9\mu + 9x_2^2 - \frac{15}{4}\right)X_3 + x_2\left(-3\mu^2 + 3\mu - 3x_2^2 + \frac{5}{4}\right)$$
$$D^{(3)}P_{X_3} = -6x_2x_3P_{X_1} - 6x_3\left(\frac{1}{2}-\mu\right)P_{X_2} - \left(\frac{3}{8}x_3(2\mu-1)\left(24x_2^2 - 6x_2 + 12\mu x_2 + 5\right)\right)X_1$$
$$+3x_2x_3\left(-2+3x_2^2\right)X_2 + \left(-9\left(\frac{1}{2}-\mu\right)^2x_3^2 + 9x_2^2x_3^2\right)X_3 + 3x_3\left(\frac{1}{2}-\mu\right)^2 - 3x_2^2x_3$$

Now we can rewrite the solution up to the third order as;

$$\mathbf{X} = \sum_{n=1}^{3}\sum_{i=1}^{3}\left\{\mathrm{M}_{i,n}X_i + \mathrm{N}_{i,n}P_{X_i} + \mathrm{C}_n^{\mathbf{X}}\right\}_{\mathbf{X}=\mathbf{X}_0,\mathbf{P}=\mathbf{P}_0}\frac{(t-t_\circ)^n}{n!}$$

$$\mathbf{P_X} = \sum_{n=1}^{3}\sum_{i=1}^{3}\left\{\mathrm{A}_{i,n}X_i + \mathrm{B}_{i,n}P_{X_i} + \mathrm{C}_n^{\mathbf{P_X}}\right\}_{\mathbf{X}=\mathbf{X}_0,\mathbf{P}=\mathbf{P}_0}\frac{(t-t_\circ)^n}{n!}$$

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where

$$M_{1,1} = -1 = -M_{2,1}^{2}$$
$$A_{1,1} = 3\eta - \frac{3}{2}x_3 + 3\mu x_3 - 6\mu x_2 - \frac{1}{4}$$
$$A_{3,1} = \frac{3}{2}x_3 - 3\mu x_3 + 3x_2x_3 + 3x_3^2 - 1$$
$$M_{2,2} = -3\left(1 - x_2^2\right)$$
$$N_{1,1} = 1 = N_{2,1}^{2}$$
$$A_{2,1} = 3x_2^2 + 3x_2x_3 - 1$$
$$B_{1,1} = -1 = -B_{0,1}^{2}$$
$$M_{1,2} = 3x_2 - 6\mu x_2 - \frac{5}{4}$$
$$M_{3,2} = \frac{3}{2}x_3\left(2x_2 - 2\mu + 1\right)$$
$$N_{1,2} = -2 = -N_{0,2}^{1}$$
$$A_{1,2} = -3x_2^2 - 3x_2x_3 + \frac{5}{4}$$
$$A_{2,2} = 3x_2 - \frac{3}{2}x_3 + 3x_3\mu + 3x_2^2 - 6\mu x_2 - \frac{5}{4}$$
$$A_{3,2} = \frac{3}{2}x_3\left(2\mu + 2x_2 - 1\right)$$
$$M_{2,3} = 6x_2^2 - \frac{13}{4}$$
$$B_{1,2} = 3x_2 - \frac{3}{2}x_3 + 3\mu x_3 - 6\mu x_2 - \frac{5}{4}$$
$$B_{2,2} = 3x_2^2 + 3x_2x_3 - 2$$
$$M_{1,3} = -\frac{1}{2}x_2\left(6x_2^2 - 7\right)$$
$$N_{1,3} = 3x_2 - 6\mu x_2 + \frac{13}{4}$$
$$A_{2,3} = 9x_2^4 + 9x_2^3x_3 - 12x_2^2 - 6x_2x_3 + \frac{13}{4}$$
$$M_{3,3} = 3x_3\left(2\mu + 2x_2 - 1\right)$$
$$N_{2,3} = 3x_2^2 - 1$$
$$A_{3,3} = -\frac{9}{4}x_3^2 - \frac{15}{4}\mu x_3 - \frac{3}{2}x_2x_3 + 9\mu x_3^2 + 9x_2^3x_3 - 18\mu x_2x_3$$
$$-9\mu^2x_3^2 + 9x_2^2x_3^2 + 18\mu^2x_2x_3 - \frac{15}{8}x_3$$
$$B_{2,3} = 6x_2 - 3x_3 + 6\mu x_3 + 3x_2^2 - 12\mu x_2 - \frac{5}{2}$$
$$B_{1,3} = -6x_2^2 - 6x_2x_3 + \frac{13}{4}$$
$$C_2^x = -\left(\frac{1}{2} - \mu\right) - x_2,$$
$$C_3^x = 1 - 2x_2 - 2\mu$$

$$C_1^{Px} = -\left(\frac{1}{2} - \mu\right) - x_3 - x_2,$$

$$C_2^{Px} = \frac{3}{2}x_3 - \mu - x_2 - 3\mu x_3 + \frac{1}{2}$$

$$C_3^{Px} = \frac{3}{4}x_3 - \frac{5}{4}\mu + \frac{1}{2}x_2 - 3\mu x_3 - 3x_2^3 + 6\mu x_2 + 3\mu^2 x_3 - 3x_2^2 x_2 - 6\mu^2 x_2 + \frac{5}{8}$$

$$A_{1,3} = \frac{15}{8}x_3 - \frac{45}{8}x_2 - \frac{15}{4}\mu x_3 - \frac{9}{4}x_2 x_3 + \frac{3}{2}x_2^2 + \frac{9}{2}x_2^3 + 18\mu^2 x_2^2 + \frac{45}{4}\mu x_2 + 9x_2^2 x_2$$
$$-18\mu x_2^2 - 9\mu x_2^3 + 9\mu x_2 x_3 - 18\mu x_2^2 x_3 - 9\mu^2 x_2 x_3 + \frac{25}{16}$$

Conclusion and Outlook

Three dimensional restricted three three body problem is constructed. The relative equilibria are obtained. An equilibruim circle is discovered namely Fawzy Equilibruim circle (in brief FEC). Infinitesimal orbits near FEC using an approach developed by Delva and Hanslmeier are derived. We obtained implicit formulas for the position and momentum vectors for n^{th} order and explicit formulas for the the same vectors for the first, second and third orders. In forthcoming works we aim to evaluate the periodic orbits near FEC in the elliptic and/or oblate restricted three body problem. Also we hope to investigate the perturbed location and stability of FEC in this framework. The same points will also be treated in the domain of the relativestic restricted three body problem.

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